ON THE RELATIONS BETWEEN THE BIFURCATION OF THE EQUILIBRIA OF CONSERVATIVE SYSTEMS AND THE STABILITY DISTRIBUTION ON THE EQUILIBRIA CURVE

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The basic works in the theory of the bifurcation of the equilibrium points are those of Poincaré [1 and 2], and Chetaev [3 and 4]. From [1 to 4] it follows: (1) The points of stability changes on the branches of the equilibria curve are points of bifurcation of the equilibrium (*); (2) the distribution of the stability on the branches of the equilibria curve in the neighborhood of the bifurcation point follows a determined law. This law has a simpler form (we shall call it "particular law of distribution of the stability") for systems with one degree of freedom and systems with n degrees of freedom for which the rank of the Hessian of the potential energy is equal to $n-1$. (See below Section 2, para $\mathcal{P}$). The potential energy is supposed to be analytic

In the derivation of statement (2), it has been assumed in [1 to 4] that the zeros of the Hessian of the potential energy (the critical points) are isolated on the branches of the equilibria curve. The statement (1) follows from [1 to 4] when a more rigorous condition is satisfied on the branch under consideration: the condition of sign change of the Hessian (see the note to the theorem 2.1).

In the present paper it is shown (Theorem 2.1) that the statement (1) is satisfied even if these conditions are not satisfied. At the same time, the condition of isolation of critical points is true for the statement (2) that of the distribution of the equilibrium points (an example is given of a potential energy for which this law is not satisfied because the indicated condition is not fulfilled, see Section 3). A derivation is made of a class of systems, important for practical cases, for which the particular law of distribution of the equilibrium points remains valid even if the condition of isolation of the critical points is not satisfied (Theorem 2.4).

Particular consideration has been given to the case in which some position of equilibrium exists for all values of the parameter α and is stable for $\alpha < \alpha_0$ where α_0 is the bifurcation value. According to the theorems of Poincaré-Chetaev, this position of equilibrium can keep or lose its stability at the point α_0, depending upon the number of branches of the curve of the equilibria which intersect at that point. It is shown (Theorem 2.2), that for the class of systems mentioned above, the stability is lost at that point, whereupon the condition of isolation of the critical points is not assumed to be satisfied and the rank of the Hessian is not assumed to be equal to $n-1$ (From there follow the

*). By points of bifurcation of the equilibria, one means the points of intersection of the branches of the equilibria curve and the limit points in the sense of Poincaré [1] (For more specific explanation see Definition 1.2 and the note at the end of Section 1).

conclusions concerning the number of branches which intersect the point considered). An analogous result has been obtained for linear systems by Ziegler [5].

In this paper we do not touch upon the important question of using the Poincaré-Chetaev theory of bifurcation of the equilibrium points. We mention Rumiantsev's paper [6] in connection with this.

1. We shall investigate conservative systems with a finite number of degrees of freedom and potential energies $\Pi(\mathbf{x},\alpha)$ depending on one real parameter $\alpha \in A$ Here $\mathbf{x} = (x^1, \ldots, x^n)$ is a vector of the coordinate space R changing in a certain region containing the two θ of space R; A represents an interval of the numerical axis.

The neighborhood of the points $\mathbf{x} \in R$, $\alpha \in A$, $(\mathbf{x}, \alpha) \in R \times A$ are denoted, respectively by $S(\mathbf{x})$, $I(\alpha)$, $O(\mathbf{x}, \alpha)$, and the ϵ - vicinity by $S^\varepsilon(\mathbf{x})$, $I^\varepsilon(\alpha)$, $O^\varepsilon(\mathbf{x}, \alpha)$.

By the curve of the equilibria B is understood the set of points of $R\times A$ satisfying Eq.

$$\text{grad}_x \Pi(\mathbf{x}, \alpha) = 0 \tag{1.1}$$

Definition 1.1 (see [4]). The point $(\mathbf{x}_0, \alpha_0)$ of the equilibria curve is said to be critical, if at this point the Hessian of the potential energy is equal to zero

$$\Delta(\mathbf{x}_0, \alpha_0) = \det\left(\frac{\partial^2 \Pi(\mathbf{x}_0, \alpha_0)}{\partial x^i \partial x^j}\right) = 0 \tag{1.2}$$

Definition 1.2 (*) The point $(\mathbf{x}_0, \alpha_0)$ of the curve of the equilibria is said to be a (real) point of bifurcation of the equilibria if for every $\epsilon > 0$ there is an α', $|\alpha_0 - \alpha'| < \varepsilon$ such that in the ϵ - neighborhood of the point $\mathbf{x}_0$ there are for $\alpha = \alpha'$ at least two positions of equilibrium $\mathbf{x}_1$, $\mathbf{x}_2$ i.e.

$$\mathbf{x}_1, \mathbf{x}_2 \in S^\varepsilon(\mathbf{x}_0), \alpha' \in I^\varepsilon(\alpha_0), \text{grad}_x \Pi(\mathbf{x}_1, \alpha') = \text{grad}_x \Pi(\mathbf{x}_2, \alpha') = 0$$

Definition 1.3 A point of bifurcation $(\mathbf{x}_0, \alpha_0)$ is said to be a limit point if in a sufficiently small neighborhood of this point for $\alpha \le \alpha_0$ $(\alpha \ge \alpha_0)$ there are no points of equilibrium different from $(\mathbf{x}_0, \alpha_0)$.

A point $(\mathbf{x}', \alpha')$ of the equilibria curve is said to be a point of stable (or unstable) equilibrium if $\mathbf{x}'$ is a point of stable (or unstable) equilibrium for $\alpha = \alpha'$.

Definition 1.4 (see [4]) We shall call points of change of stability on some branch C of the equilibria curve, the points located at the limits of the regions of stability and instability on C (it is assumed that those domains are the intervals of the branch C).

By a branch of the equilibria curve is meant the smooth curve $C \subset B$. From the definition of the equilibria curve its branches are real. Furthermore, at times the terms "imaginary branch" and "semibranch, emanating from some point" are used [8].

We shall make a few remarks concerning the meaning of "point of bifurcation of the equilibria".

Definition 1.2 is equivalent to the statement that the point $(\mathbf{x}_0, \alpha_0)$ appears as a branch point of the real solution of Eq. (1.1). For an analytic $\Pi(\mathbf{x}, \alpha)$ such a point appears either as a point of intersection of the equilibria curves or a limit point in the sense of Poincare [1] (**).

*) Compare with the definition of the branch point of an operator, depending on a parameter [7].

**) In both cases, the point $(\mathbf{x}_0, \alpha_0)$ belongs at least to two single parameter (parameter α) continuous families of real forms of equilibrium [1 to 4]. In the second case they form a branch, tangent to the hyperplane $\alpha = \alpha_0$ located in the domain $\alpha \le \alpha_0$ $(\alpha \ge \alpha_0)$ and intersecting at least one imaginary branch.

Chetaev [4], p.52) calls the point of bifurcation of the equilibrium a point satisfying the condition (1.2), i.e. he uses this term as a synonim of the term "critical point" also accepted by him.

A point satisfying (1.2), is by definition a branch point for the real solutions of Eq.(1.1). Moreover it is pointed out in [1 and 4] that the condition (1.2) is necessary but not sufficient for branching. It is possible to have particular cases for which for an analytical $\Pi(\mathbf{x},\alpha)$ a point satisfying (1.2) is neither a point of intersection of the real branches nor a limit point.

For instance, such is the point (O, O) for the potential energy $\Pi(x, \alpha) = x^4 + \alpha^2 x^2$. The equilibria curve consists here of a single branch, the curve $x = 0$. The point (O, O) is the only critical point.

In this case, there is an imaginary branching at the point (O, O); this always occurs in particular cases of systems, satisfying the condition of isolation of critical points (systems of this type are considered in [1 to 4]).

When this condition is violated, the critical point may have neither a real nor an imaginary branching, as, for instance in the trivial case of the potential energy $\Pi = \alpha x^4$ $(\alpha > 0)$ Here the Hessian of the potential energy is identically equal to zero on the equilibria curve $x = 0, \alpha > 0$.

Thus Definition 1.2 is equivalent to Chetaev's definition with an accuracy up to the particular cases mentioned.

In the first of the examples shown above the point (O, O) is critical, but is not a point of bifurcation in the sense of Definition 1.2. In the second case all points of the straight line $x = 0$, $\alpha > 0$ are such points.

Since in the present paper the imaginary branches of the equilibria curve are not considered, and it is not assumed that the condition of isolation of the critical points is satisfied, Definition 1.2 turns out to be convenient.

2. 1°. Theorem 2.1 Let $(\mathbf{x}_0, \alpha_0)$ be a point of stability change on some branch of the equilibria curve. For this point to be a point of bifurcation, in the case of a conservative system with a finite number of degrees of freedom and a potential energy $\Pi(\mathbf{x},\alpha)$ analytic in $\mathbf{x}$, it is sufficient that $\mathrm{grad}_\mathbf{x}\Pi(\mathbf{x},\alpha)$ be continuous on the set $(\mathbf{x},\alpha)$ (for this purpose, it is obviously sufficient that $\Pi(\mathbf{x},\alpha)$ be analytical on $(\mathbf{x},\alpha)$).

Note. With the additional condition that "on the branch C the Hessian of the potential energy which becomes zero at the point $(\mathbf{x}_0, \alpha_0)$ changes its sign at that point" Theorem 2.1 follows from [1 to 4] (see, for instance, the theorem on the number of branches [4], p. 53). Whereupon it is noted in [1] that if the Hessian is equal to zero in C or does not change its sign at the critical point $(\mathbf{x}_0, \alpha_0)$, then $(\mathbf{x}_0, \alpha_0)$ may not be a point of bifurcation. From Theorem 2.1 it follows that this possibility is not realized if $(\mathbf{x}_0, \alpha_0)$ is a point of stability change

Proof. The following lemma is used in the proof.

Lemma. Given the single parameter family of real functions $\Pi(\mathbf{x},\alpha)$ with the vector argument $\mathbf{x} \in R$ and the scalar parameter $\alpha \in (-\infty, +\infty)$, satisfying the following conditions:

(a) $\mathrm{grad}_\mathbf{x}\Pi(\mathbf{x},\alpha)$ is continuous on the set $(\mathbf{x},\alpha)$;

(b) $\Pi(\theta, \alpha) \equiv \mathrm{grad}_x \Pi(\theta, \alpha) \equiv 0$;

(c) For each fixed $\alpha < 0$ the zero θ of the space R is an exact minimum point of the function $\Pi(\mathbf{x},\alpha)$ i.e. there exists a neighborhood $S(\theta)$ depending on α and such that

$\Pi(\mathbf{x},\alpha) > 0$ for $\mathbf{x} \in S(0)$, $\mathbf{x} \neq 0$, $\alpha < 0$;

(d) For any $\epsilon > 0$ there exists a pair $(\mathbf{x}^\circ, \alpha^\circ) \in O^\epsilon(\theta, 0)$, $\mathbf{x}^\circ \neq \theta$ such that $\Pi(\mathbf{x}^\circ,\alpha^\circ) = 0$.

Then for each $\epsilon > 0$ there exists

$$(\mathbf{x}', \alpha') \in O^\epsilon(0, 0), \qquad \mathbf{x}' \neq 0, \qquad \mathrm{grad}_x\, \Pi(\mathbf{x}', \alpha') = 0 \tag{2.1}$$

The proof of the Lemma will be given for a two-dimensional space R, but with an accuracy up to the terminology [9] it is also valid for arbitrary dimensions. We shall designate by $K(t,\alpha)$ the level line of the function $\Pi(\mathbf{x},\alpha)$(α fixed) for which this function takes on the value t.

Let us consider the case for which the condition (d) is satisfied for $\alpha < 0$(the other possibilities can be considered analogously). In such a case it is possible to find a sequence $\{\alpha_n\}$, $\alpha_n \to 0$, $\alpha_n < 0$, for which there exists a sequence $\{\mathbf{x}_n\}$ such that

$$\mathbf{x}_n \to \theta \quad \text{for} \quad \alpha_n \to 0, \quad \alpha_n < 0; \quad \Pi(\mathbf{x}_n, \alpha_n) = 0$$

Let us assume that the Lemma is wrong. Then there exist neighborhoods $S^*(\theta)$, $I^*(0)$ such that

$$\mathrm{grad}_x\, \Pi(\mathbf{x}, \alpha) \neq 0, \qquad \mathbf{x} \in S^*(\theta), \qquad \mathbf{x} \neq 0, \qquad \alpha \in I^*(0) \tag{2.2}$$

(for the sake of convenience we shall assume that (2.2) is also satisfied on the boundary of the neighborhood $S^*(\theta)$).

Let us fix some α_n from the sequence $\{\alpha_n\}$ determined above and the $\mathbf{x}_n \in \{\mathbf{x}_n\}$ which correspond to it.

On the basis of (2.2) and the condition (c) of the Lemma, in a sufficient vicinity of the point θ, the level lines of the function $\Pi(\mathbf{x},\alpha_n)$ for fixed α_n represent simple closed curves which encircle the point θ, do not include stationary points of the function $\Pi(\mathbf{x}, \alpha_n)$ and have inside their domain, level lines of the specified type only. We shall denote by Ω_n the domain swept by that family (the subscript n indicates that α_n is fixed).

The following is easy to check:

1) from any point of the domain Ω_n there is a line of steepest descent γ_n in θ;

2) the boundary of the domain Ω_n belongs to the level line $K(t_n,\alpha_n)$ of the dividing [9] point θ and $\mathbf{x}_n$, whereby $t_n \to 0$ for $\alpha_n \to 0$. It is evident that

$$\sup \Pi(\mathbf{x}, \alpha_n) = t_n, \qquad \mathbf{x} \in \Omega_n$$

Since the neighborhood $S^*(\theta)$ does not contain stationary points of the function $\Pi((\mathbf{x},\alpha_n)$ the domain Ω_n intersects with the boundary of that domain (in the converse case the family of level lines considered above could have been extended). We shall represent by b_n one of the intersection points. It is evident that

$$\sup_{\mathbf{x}\in\gamma_n} |\Pi(\mathbf{x}, \alpha_n)| \to 0 \quad \text{for} \quad \alpha_n \to 0$$

On the basis of the finite dimensions of the space R, there is at least one point $\mathbf{x}^* \neq \theta$, in $S(\theta)$, any neighborhood of which intersects with an infinite number of curves γ_n; all such points belong to the level line $K(0, 0)$ whereby the entire sequence $\{\gamma_n\}$ is contained in any neighborhood of the level line. From there on, taking into consideration the condition (a) of the Lemma and also the fact that the direction γ_n at each point coincides with the direction $\mathrm{grad}_x\, \Pi(\mathbf{x}, \alpha_n)$ at the same point, we get

$$\mathrm{grad}_x\, \Pi(\mathbf{x}^*, 0) = 0$$

which contradicts the assumption (2.2) and therefore proves the Lemma.

It is not difficult to prove that the following is true if the conditions (a),(b) of the Lemma are satisfied, and the conditions (c),(d) are replaced by:

(c') for $\alpha < 0$, there exists a neighborhood $S(\theta)$ such that $\Pi(\mathbf{x}, \alpha) \geqslant 0$, $\mathbf{x} \in S(\theta)$ ($S(\theta)$ depends on α,

(d') for $\alpha > 0$ in each $S(\theta)$ there exists $\mathbf{x}' \neq \theta, \Pi(\mathbf{x}', \alpha) \leq 0$, then (2.1) is satisfied.

We shall show that if this result is taken into consideration, Theorem 2.1 follows from the Lemma. It is possible to confine oneself to the case in which the line $\mathbf{x} = \theta$ belongs to the equilibria curve and the point of stability change $(\theta, 0)$ located on that branch is considered. (Indeed, if $(\mathbf{x}_0, \alpha_0)$ is a point of stability change on some arbitrary branch C, then either $(\mathbf{x}_0, \alpha_0)$ is a limit point or in the neighborhood of that point the branch C has an explicit representation $\mathbf{x} = \mathbf{x}(\alpha)$ and by an appropriate change of variables it is possible to reduce the problem to the case pointed out earlier). Let the branch $\mathbf{x} = \theta$ be stable for $\alpha < 0$ and unstable for $\alpha > 0$ (α sufficiently small). Let us also take $\Pi(\theta, \alpha) \equiv 0$. Then for the potential energy $\Pi(\mathbf{x}, \alpha)$ the relation (2.1) of the Lemma is equivalent to the fact that $(\theta, 0)$ is a point of bifurcation, i.e. a confirmation of Theorem 2.1. But for the potential energy $\Pi(\mathbf{x}, \alpha)$ the conditions (a),(b) of the Lemma are obviously satisfied, and the conditions (c'),(d') follow from [10] (Section 5) and the Theorems of Lagrange, Lejeune-Dirichlet (note that for all sufficiently small fixed α, the position of equilibrium $\mathbf{x} = \theta$ can be considered as isolated, since in the opposite case (2.1) is known to be satisfied). Theorem 2.1 is proved.

In two particular cases of conservative systems, for systems with one degree of freedom and a potential energy analytic in $\mathbf{x}$, and for systems with many degrees of freedom having for potential energy a form in $\mathbf{x}$, Theorems 2.1 can be somewhat strengthened: in these sytems it is sufficient to require the continuity of $\Pi(\mathbf{x}, \alpha)$ with respect to α for fixed values of $\mathbf{x}$.

2. Let the line $\mathbf{x} = \theta$ belong to the equilibrium curve B i.e.

$$\mathrm{grad}_x\ \Pi(\theta, \alpha) \equiv 0 \tag{2.3}$$

and let the point (θ, α_0) be a point of bifurcation, whereby the equilibrium $\mathbf{x} = \theta$ is stable for $\alpha < \alpha_0$ or $\alpha > \alpha_0$.

From [1 to 4] it is easy to obtain that if (θ, α_0) is an isolated critical point, the minor Δ_1 of the first angular element of the Hessian is not equal to zero at the point (θ, α_0) and if in the domain $x^1 > 0$ ($x^1 < 0$) there is an odd number of semibranches passing through the point (θ, α_0), then that point is a point of stability change on the branch $\mathbf{x} = \theta$; if the number is even then it lies in the domain of stability of that branch. If the condition $\Delta_1(\theta, \alpha_0) \neq 0$ is not fulfilled or if the critical point (θ, α_0) is not isolated, then it is not known whether it appears as a point of stability change.

We shall limit ourselves now to the investigation of systems, the potential energy of which has the form (*)

$$\Pi(\mathbf{x}, \alpha) = U(\mathbf{x}) - f(\alpha) V(\mathbf{x}) \tag{2.4}$$

*) Compare with the special form of the potential energy in [5]. The form (2.4) is obtained for instance for the potential energy of conservative elastic systems with n degrees of freedom, in which the parameter α is a force parameter and $U(\mathbf{x})$ is the potential energy of deformation [5]. It is, in general, possible to reduce to a similar form the expression of Routh's potential in problems of the stability of the stationary motions of conservative non-gyroscopically coupled systems. Here using the theory of Poincaré-Chetaev, one has as parameters the generalized impulses of the cyclic coordinates [6].

where $f(\alpha)$ is a continuous monotonic function, $U(\mathbf{x})$, $V(\mathbf{x})$ are analytic functions, satisfying the following property: if there exists an equilibrium position $\mathbf{x}_0$ which is kept for all values of the parameter α, then the increment at the point $\mathbf{x}_0$ of at least one of the functions, U or V is sign-definite in the neighborhood of that point.

Theorem 2.2. Let us assume (2.3) is satisfied and the potential energy has the form (2.4). Then if (θ,α_0) is a point of bifurcation of the equilibria and the equilibrium $\mathbf{x}=\theta$ is stable for $\alpha<\alpha_0$ or $\alpha>\alpha_0$ then (θ,α_0) is a point of stability change on the branch $\mathbf{x}=\theta$ (i.e. the point (θ,α_0) cannot lie inside the domain of stability of the branch $\mathbf{x}=\theta$).

Proof. Let (θ,α_0) be a point of bifurcation; $\mathbf{x}=\mathbf{x}(\alpha)$ is the branch C of the equilibria curve, which passes through the point (θ,α_0), and does not coincide with the line $\mathbf{x}=\theta$ nor is orthogonal to it (if C is orthogonal to the axis A, the theorem is obvious); C_x, C_α are, respectively, the projections of C on R and A. Without loss of generality it can be assumed that $U(\theta)=V(\theta)=0$, and consequently $\Pi(\theta,\alpha)\equiv 0$.

The proof of the theorem is based on the statement

$$\lim \frac{U(\mathbf{x})}{V(\mathbf{x})} = f(\alpha_0) \quad \text{for } \mathbf{x}\to\theta, \quad \mathbf{x}\in C_x \tag{2.5}$$

which follows if the sign definiteness of $U(\mathbf{x})$ ($V(\mathbf{x})$) in the neighborhood of zero is taken into consideration, from the identity

$$\operatorname{grad} U(\mathbf{x}) - f(\alpha)\operatorname{grad} V(\mathbf{x}) \equiv 0, \qquad (\mathbf{x},\alpha)\in C \tag{2.6}$$

Let us show that (2.5) follows from (2.6). Let us introduce the notation

$$U_i(\mathbf{x}) = \frac{\partial U(\mathbf{x})}{\partial x^i}, \qquad V_i(\mathbf{x}) = \frac{\partial V(\mathbf{x})}{\partial x^i}$$

It is easy to verify that there are numbers i (let $i=1,2,\ldots m, m\le n$) such that in a sufficiently small neighborhood of the point (θ,α_0)

$$U_i(\mathbf{x})\neq 0, \quad V_i(\mathbf{x})\neq 0 \quad \text{for } \mathbf{x}\neq\theta, \quad \mathbf{x}\in C_x \quad (i=1,\ldots,m)$$
$$U_i(\mathbf{x})\equiv V_i(\mathbf{x})\equiv 0 \quad \text{for } \mathbf{x}\in C_x \quad (i=m+1,\ldots,n)$$

It follows, taking (2.6) into account that

$$\lim \frac{U_i(\mathbf{x})}{V_i(\mathbf{x})} = f(\alpha_0) \quad \text{for } \mathbf{x}\to\theta, \quad \mathbf{x}\in C_x \quad (i=1,\ldots,m)$$

Since $U(\theta)=V(\theta)=0$, then, using the sign definiteness of $U(\mathbf{x})$ in the neighborhood of zero we get

$$\lim_{\alpha\to\alpha_0}\frac{U(\mathbf{x}(\alpha))}{V(\mathbf{x}(\alpha))} = \lim_{\alpha\to\alpha_0}\frac{\Sigma U_i dx^i(\alpha)}{\Sigma V_i dx^i(\alpha)} = f(\alpha_0)\lim_{\alpha\to\alpha_0}\frac{\Sigma U_i dx^i(\alpha)}{\Sigma U_i dx^i(\alpha) + o(\Sigma U_i dx^i(\alpha))}$$

Here the summation with respect to i extends from 1 to m. From the last equation (2.5) follows. On the basis of (2.5) and the monotonic behavior of $f(\alpha)$ the point (θ,α_0) does not appear as a point internal to the domain on the line $\mathbf{x}=\theta$. This proves the Theorem.

Theorem 2.3 If for a conservative system having a finite number of degrees of freedom the potential energy is a form in x monotonously and continuously decreasing in α for fixed **x**, then all the bifurcation points are inside or on the boundary of an unstable trivial equilibrium position (which, obviously, exists for all α).

Let us note that from the Expression (2.4) the monotonic behaviour of $\Pi(\mathbf{x},\alpha)$ in α follows (one of the requirements of Theorem 2.3). Thus, for arbitrary systems unlike those systems for which the potential energy appears as a form (in particular for linear systems), this requirement alone is not sufficient (see Example 3.2).

It is easy to prove Theorem 2.3 by using the following property of homogeneous functions: if $f(\mathbf{x})$ is a homogeneous function and $f(\mathbf{x}^*) = 0$, $\mathbf{x}^* \neq \theta$, then $f(\mathbf{x}) \equiv 0$, at least on the path, connecting the points $\mathbf{x}^*$ and θ. The proof is analogous to the proof of the corresponding theorem by Ziegler [5] which concerns linear systems.

3°. Let $(\mathbf{x}_0, \alpha_0)$ be a point of bifurcation of the equilibria, and let us assume that at that point the rank of the Hessian of the potential energy is equal to n-1. Let for definiteness, the minor

$$\Delta_1 = \det\left(\frac{\partial^2 \Pi(\mathbf{x}_0, \alpha_0)}{\partial x^i \partial x^j}\right) \neq 0 \qquad (i, j = 2, \ldots, n) \tag{2.7}$$

be different from zero.

In that case a system with n degrees of freedom can be reduced to a system with one degree of freedom in the "auxiliary space" (x^1, α) [1 to 4]. Such systems were considered by Poincaré [1 and 2]. Theorems on the distribution of the stability were obtained by Chetaev [3 and 4] without the limitations (2.7).

Let us consider some information obtained from [1 to 4] (see also [8]).

If (2.7) is satisfied, then Eqs.

$$\frac{\partial \Pi(\mathbf{x}, \alpha)}{\partial x^2} = 0, \ \ldots, \qquad \frac{\partial \Pi(\mathbf{x}, \alpha)}{\partial x^n} = 0 \tag{2.8}$$

determine in the neighborhood of the point $(\mathbf{x}_0, \alpha_0)$ the surface

$$\mathbf{x} = \mathbf{x}(x^1, \alpha) \quad (x^2 = x^2(x^1, \alpha), \ldots, \ x^n = x^n(x^1, \alpha))$$

which intersects with hyperplane $\alpha = \alpha_0 - \epsilon$ (ϵ sufficiently small) along the simple (connected, without multiple points) curve L: $\mathbf{x} = \mathbf{x}(x^1, \alpha_0 - \epsilon)$ (if (2.7) is not satisfied, then L consists of a few simple pieces). The equilibria curve B lies on the surface (2.8). Its branches, passing through the point $(\mathbf{x}_0, \alpha_0)$ intersect with L at a few points $P_1, \ldots, P_p$ numbered along the increasing coordinates x^1. The potential energy

$$\Pi^*(x^1, \alpha) = \Pi(x^1, x^2(x^1, \alpha), \ldots, x^n(x^1, \alpha), \alpha)$$

determines in the auxiliary space (x^1, α) the equilibria curve B^*

$$\frac{\partial \Pi^*(x^1, \alpha)}{\partial x^1} = 0$$

which appears as the projection of the curve B on the surface (x^1, α). Thereby the points $P_1, \ldots, P_p$ are projected into the points $P_1^*, \ldots, P_p^*$ of intersection of the curves B^* with the straight line $\alpha = \alpha_0 - \epsilon$; and the curve L is projected on the straight line $\alpha = \alpha_0 - \epsilon$. Proceeding with analogous constructions for the hyperplane $\alpha = \alpha_0 + \epsilon$ we get the points $Q_1, \ldots, Q_q$ and $Q_1^*, \ldots, Q_q^*$.

Let at the point of bifurcation $(\mathbf{x}_0, \alpha_0)$ the zero of the Hessian be isolated on the equilibria curve, i.e. in some neighborhood of that point

$$\Delta(\mathbf{x}, \alpha) \neq 0 \quad \text{for} \quad (\mathbf{x}, \alpha) \neq (\mathbf{x}_0, \alpha_0), \qquad (\mathbf{x}, \alpha) \in B \tag{2.9}$$

and let the matrix (Δ_1) of the minor (2.7) be positive-definite at the point $(\mathbf{x}_0, \alpha_0)$. We shall denote it by

$$(\Delta_1) > 0 \tag{2.10}$$

(for this purpose, when the condition (2.7) is satisfied it is sufficient to have at least one stable semibranch passing through the point (x_0, α_0). Then the following relations are valid

$$[P_i] = [P_i^*] \tag{2.11}$$

$$[P_i] = -[P_{i+1}] \tag{2.12}$$

$$\sum_1^p [P_i] = \sum_1^q [Q_j] \tag{2.13}$$

$$[P_1] = [Q_1], \quad [P_p] = [Q_q] \tag{2.14}$$

where $[P] = +1$, if the point P is stable, $[P] = -1$ if the point P is unstable.

The relation (2.11) indicates that to the stable points of the system investigated correspond stable points of the auxiliary system, and to the unstable ones correspond unstable points (the first is satisfied even without (2.10)). The relation (2.12) means that the stable and unstable points P_i alternate on the curve L as the points P_i^* which correspond to them on the curve $\alpha = \alpha_0 - \epsilon$. The relation (2.14) means that the first points P_1 and Q_1 have the same character of stability: the same thing holds for the last points P_p and Q_q.

Let us note that if (2.10) is not satisfied, then instead of interchanging the stable and unstable points P_i (along (2.12)) the order of instability [4] is changed (then (2.12) is satisfied in the auxiliary system).

For the relations (2.11) to (2.14) the condition (2.9) is essential (see Section 3). Similarly, it is shown further on that for a potential energy of the form (2.4) these relations remain valid even without the condition (2.9). In other words the following theorem is valid.

Theorem 2.4 Let (x_0, α_0) be a point of bifurcation of the equilibria and assume the potential energy has the form (2.4). Then for systems with one degree of freedom the relations (2.12) to (2.14) are valid; for systems with n degrees of freedom, satisfying (2.10) the relations (2.11) to (2.14) are valid.

Proof. Let us consider first a system with one degree of freedom. It is simple to see that the relations (2.12) to (2.14) are satisfied if $\Pi(x, \alpha)$ considered on the straight line $\alpha = \alpha_0 - \epsilon$ ($\alpha = \alpha_0 + \epsilon$), has at the points of intersection of that straight line and the branches of the equilibria curve extremas in x (i.e. for $\Pi(x, \alpha_0 - \epsilon)$ the stationary points do not appear as inflexion points, which happens, in particular if the Hessian is equal to zero). But at the points of the branch $x = x_0$ (if such a branch exists) the potential energy (2.4) has extrema in x: this follows from (2.5) by taking into consideration the sign definiteness of the increment $U(x) - U(x_0)$.

However, on the branches which do not coincide with $x = x_0$ the Hessian $\partial^2 \Pi / (\partial x)^2$ of the potential energy (2.4) is not equal to zero (and, consequently, $\Pi(x, \alpha)$ has also an extremum in x). Indeed, let on some branch $x = x(\alpha)$ the identity

$$\frac{\partial^2 \Pi(x, \alpha)}{(\partial x)^2}\bigg|_{x=x(\alpha)} \equiv 0 \tag{2.15}$$

be satisfied

Since $\partial \Pi / \partial x \equiv 0$ on the branch $x = x(\alpha)$ then

$$\frac{\partial^2 \Pi(x, \alpha)}{(\partial x)^2}\bigg|_{x=x(\alpha)} \frac{dx(\alpha)}{d\alpha} + \frac{\partial^2 \Pi(x, \alpha)}{\partial x\, \partial \alpha}\bigg|_{x=x(\alpha)} \equiv 0$$

By virtue of (2.15) we get

$$\left.\frac{\partial^2 \Pi\,(x,\ \alpha)}{\partial x\, \partial \alpha}\right|_{x=x(\alpha)} \equiv 0 \qquad (2.16)$$

If the potential energy has the form (2.4) and the branch $x = x(\alpha)$ does not coincide with $x = x_0$, then from (2.16) it follows that $dV / dx \equiv 0$, which contradicts (2.4). The theorem is proved for the case of one degree of freedom.

Let us consider a system witn n degrees of freedom and satisfying (2.10). If $\Pi(x, \alpha)$ has the form (2.4), then $\Pi^*(x^1, \alpha)$ has also a similar form. Thus, for the proof of Theorem (2.4) it is sufficient to show that independently from the condition (2.9), to stable points P of the curve L of the hyperplane $\alpha = \alpha_0 - \epsilon$ correspond stable points P^* on the straight line $\alpha = \alpha_0 - \epsilon$ of the auxiliary space (x^1, α) and conversely, to the stable points P^* correspond stable points P (see the notations above) i.e. it is sufficient to prove (2.11). Let us assume $\Pi(P) = \Pi^*(P^*) = 0$.

If in the neighborhood $O(P)$ of the point P

$$\Pi\,(x,\ \alpha_0 - \varepsilon) > 0 \quad \text{for } x \in L, \qquad (x,\ \alpha_0 - \varepsilon) \neq P$$

we shall say that Π is positive definite ("$\Pi > 0$"), in $O(P)$ on L (this is equivalent to the positive definiteness in x^1 of the function Π^* in $O(P^*)$). It is evident that if $\Pi > 0$ in $O(P)$ along all the variables $x^1, \ldots, x^n$ on the hyperplane $\alpha = \alpha_0 - \epsilon$, then $\Pi > 0$ in $O(P)$ on L. Conversely, let $\Pi > 0$ in $O(P)$ on L. Since $x^1 \neq$ const on the curve L, and on the basis of (2.10), $\Pi > 0$ in $O(P)$ in the variables $x^2, \ldots, x^n$, then $\Pi > 0$ in $O(P)$ in all the variables $x^1, \ldots, x^n$. Then, the validity of the relation (2.11) follows from (2.4). The Theorem is proved.

C o n s e q u e n c e s of Theorems 2.2 and 2.4. If the potential energy has the form (2.4) and the equilibria curve contains the branch $x = x_0$, stable for $\alpha < \alpha_0$ ($\alpha > \alpha_0$), where (x_0, α_0) is a point of bifurcation, then in sytems with one degree of freedom in the half plane $x > x_0$, an odd number of semi branches of the equilibria curve pass through (x_0, α_0); in the systems with n degrees of freedom, satisfying (2.10) in the domain $x^1 > x_0^1$ there is an odd number of branches of the equilibria curve passing through (x_0, α_0).

3. We shall give two examples of potential energy showing that for an arbitrary potential energy, the law of the stability distribution can be broken if the law (2.9) of isolation of critical points is not satisfied.

E x a m p l e 3.1 Let the potential energy have the form (one-dimensional case)

$$\Pi\,(x,\ \alpha) = 2\,\alpha^2 x^2 + {}^8/_3\,\alpha x^3 + x^4 \qquad (3.1)$$

It is easily verified that the equilibria curve on the $(x\alpha)$ plane consists of the lines $x = 0$ and $x + \alpha = 0$. The Hessian of the potential energy on the line $x = 0$ has the form $\Delta\,(\alpha) = 4\alpha^2$, and on the line $x + \alpha = 0$ the form $\Delta(\alpha) \equiv 0$. The branch $x = 0$ is entirely stable, and the branch $x + \alpha = 0$ entirely unstable (with the exception of the point $(0, 0)$). The point $(0, 0)$ is the only bifurcation point. Thus the relation (2.14) is violated. Furthermore, the equilibrium $x = 0$ does not lose its stability at the bifurcation point $(0, 0)$ which violates one of the laws of Poincaré-Schwartzschild (see for instance [8]). We notice that on the branch $x = 0$, the Hessian is not identically equal to zero, thus to violate this rule, it may be sufficient to violate the conditions (2.9) just on one branch. A trivial example of violation of the relation (2.12) (the law of the stability change) is obtained from (3.1), by multiplying the right-hand side by -1. In that case the two branches $x = 0$ and $x + \alpha = 0$ are unstable.

E x a m p l e 3.2 Let the potential energy have the form

$$\Pi(x, \alpha) = \begin{cases} 2\alpha^2x^2 + {}^8/_3\alpha x^3 + x^4 & \text{for } \alpha \leqslant 0 \\ (1-\alpha)x^4 & \text{for } \alpha > 0 \end{cases}$$

It is easily verified that the function $\Pi(x,\alpha)$ and its derivatives in x, are continuous in α, whereby $\Pi(x,\alpha)$ is monotonically decreasing in α. The equilibria curve consists of the lines $x = 0$ and $\alpha = 1$ and the half-line $x + \alpha = 0, \alpha \leq 0$. The Hessian of the potential energy on the negative semi-axis A is equal to $4\alpha^2$, on the positive semi-axis A and on the branch $x + \alpha = 0, \alpha \leq 0$ it is identically equal to zero. The branch $x = 0$ is stable for $\alpha < 1$ and unstable for $\alpha > 1$. The branch $x + \alpha = 0, \alpha \leq 0$ is unstable. Thus, here as in Example 3.1 the bifurcation point $(0,0)$ is inside the domain of stability of the branch $x = 0$ in spite of the monotonous decrease of $\Pi(x,\alpha)$ in α (the other bifurcation point (0,1) is a point of stability change on that branch). This violates the same assumptions that were violated in Example 3.1 and, furthermore, the relations (2.13), and also the formula giving the number of real branches intersecting in the bifurcation point [4] (p. 53) (to be more precise this formula loses its meaning at the point $(0,0)$).

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